Year 12 Mathematics Specialist 2019

Test Number 5:

Rates of Change and Differential Equations

Resource Free

Name: ______SOLUTIONS_____________________ **Teacher: DDA**

Marks: **28**

Time Allowed: **25 minutes**

Instructions: No notes or calculators. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 1 **[2 marks]**

What is $\frac{dy}{dx}$ for $3y^2 - 5x^3 = 4x$?

$$6y\frac{dy}{dx} - 15x^2 = 4$$

$$\frac{dy}{dx} = \frac{4 + 15x^2}{6y}$$

Question 2 **[2 marks]**

Given $y = 4x^2 + 5x$ and $\frac{dx}{dt} = 2$ when $x = 1$, find $\frac{dy}{dt}$ when $x = 1$.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\frac{dy}{dt} = (8x + 5).2$$

$$\frac{dy}{dt} = 13 \times 2 = 26$$

Question 3 **[1 marks]**

What is the general solution to $\frac{dy}{dt} = 30(2t + 1)^2$?

$$y = 5(2t + 1)^3 + c$$

Question 4 **[4 marks]**

A train leaving the station has its acceleration given by $1.8t - 0.21t^2$ for the first 10 seconds of motion. What is the speed and distance moved after 10 seconds?

$$t = 0, x = 0, v = 0$$
$$v = 0.9t^2 - 0.07t^3 + c, c = 0$$
$$v = 0.9t^2 - 0.07t^3$$
$$x = 0.3t^3 - \frac{0.07t^4}{4} + c, c = 0$$
$$x = 0.3t^3 - 0.0175t^4$$
$$\therefore t = 10, v = 90 - 70 = 20m/s$$
$$x = 300 - 175 = 125m$$

Minus 1 mark if initial conditions not considered to produce constants equalling zero.

Question 5 **[4 marks]**

Solve $\frac{dy}{dx} = (2x - 5)(y + 3)$.

$$\int \frac{1}{y+3} dy = \int 2x - 5 dx$$
$$\ln|y+3| = x^2 - 5x + c$$
$$|y+3| = ke^{x^2-5x}, k \in$$
$$x < -3, -y - 3 = ke^{x^2-5x} \Rightarrow y = -3 - ke^{x^2-5x}$$
$$x > -3, y + 3 = ke^{x^2-5x} \Rightarrow y = -3 + ke^{x^2-5x}$$
$$\Rightarrow y = -3 + ke^{x^2-5x} \forall x \in \quad : x \neq -3, k \in$$

Question 6 **[3 marks]**

Consider the differential equation $\frac{dy}{dx} = x + y - 2$. Let $y = g(x)$ be the solution to the differential equation with initial condition $g(-1) = k$ where k is a constant. Euler's method, starting at $x = -1$ with step size 1, gives the approximation $g(2) \approx -1$. Find the value of k.

x	y	y'	δy
-1	k	$-1 + k - 2 = k - 3$	$(k-3).1 = k - 3$
0	$2k - 3$	$0 + 2k - 3 - 2 = 2k - 5$	$(2k-5).1 = 2k - 5$ ✓
1	$4k - 8$	$1 + 4k - 8 - 2 = 4k - 9$	$4k - 9$
2	$8k - 17$ ✓		

$$8k - 17 = -1$$
$$8k = 16$$
$$k = 2$$
✓

No mark for a fluke answer of 2 or an answer of 2 without relevant reasoning.

Question 7 **[5 marks]**

Find the equation of the tangent to the curve $e^{y} + y\ln x = 3$ at the point $P(1, \ln 3)$.

$$e^{y}\frac{dy}{dx} + 1\frac{dy}{dx}.\ln x + \frac{1}{x}y = 0$$

$$e^{\ln 3}\frac{dy}{dx} + 1\frac{dy}{dx}.\ln 1 + \frac{1}{1}\ln 3 = 0$$

$$3\frac{dy}{dx} + 0 + \ln 3 = 0$$

$$\frac{dy}{dx} = -\frac{\ln 3}{3}$$

$$y - \ln 3 = -\frac{\ln 3}{3}(x-1)$$

$$3y = -(\ln 3)x + 4\ln 3$$

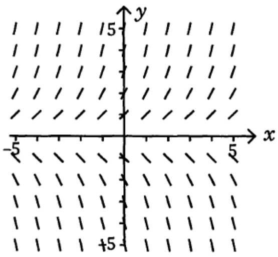
Question 8 **[3 marks]**

The three slope fields shown below have their associated differential equations in the following list. Choose the correct equation for each slope field.

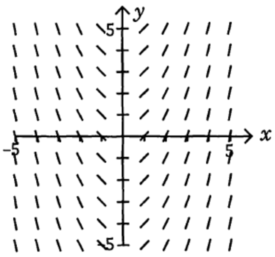
$\frac{dy}{dx} = 2y$ $\frac{dy}{dx} = y$ $\frac{dy}{dx} = x$

$\frac{dy}{dx} = (x-2)(y-3)$ $\frac{dy}{dx} = \frac{x-3}{y-2}$ $\frac{dy}{dx} = (x-3)(y-2)$

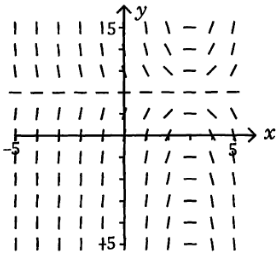
Slope field A.



Slope field B.



Slope field C.

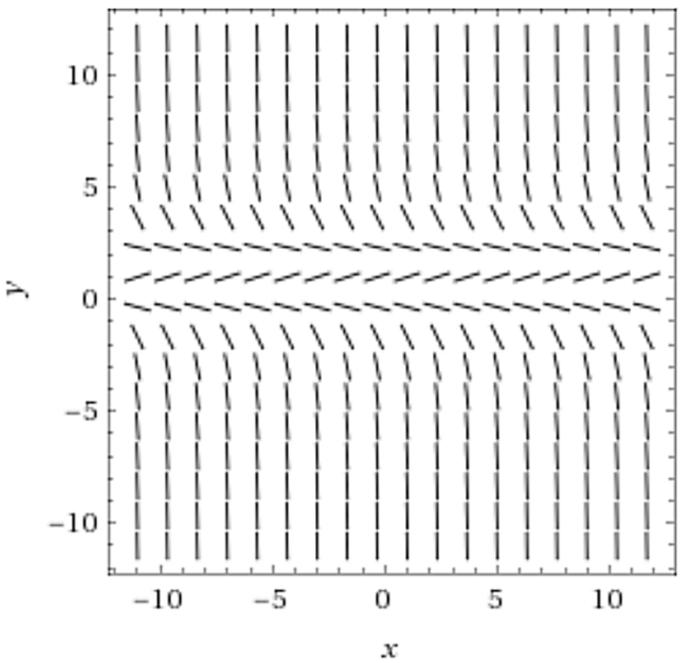


$\frac{dy}{dx} = y$ $\frac{dy}{dx} = x$ $\frac{dy}{dx} = (x-3)(y-2)$

Question 9 **[2 marks]**

Would the differential for the slope field below be of the form $\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = f(y)$?

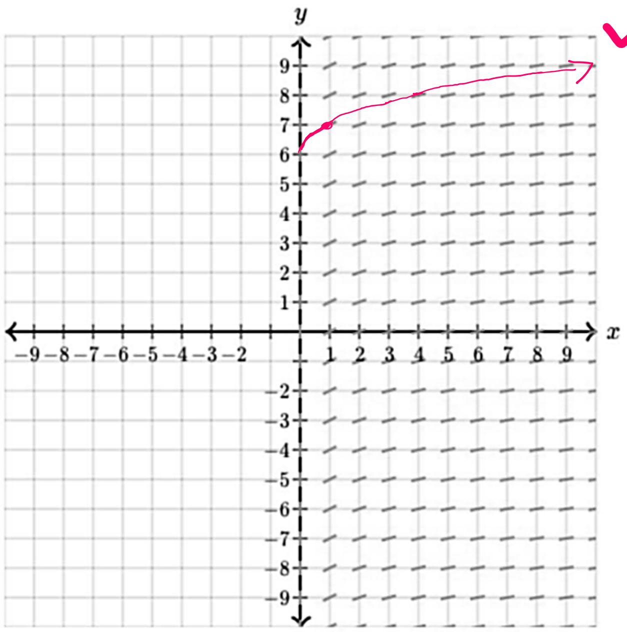
Give a reason for your answer. The slope is independent of x. (For each y-value the slope is constant across all x-values.)



Question 10 **[2 marks]**

For the slope field below, draw in the particular solution given that the initial condition (1,7).

What form would the solution to this equation be in?



Actual solution.

$y = \sqrt{x} + 6$

Accept $y = a\sqrt{x} + b$

Year 12 Mathematics Specialist 2019

Test Number 5:

Rates of Change and Differential Equations

Resource Rich

Name: ___SOLUTIONS_________________________ **Teacher: DDA**

Marks: **21**

Time Allowed: **20 minutes**

Instructions: You are permitted 1 A4 page of notes and your calculator. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 11 **[4 marks]**

A particle with SHM is moving with a velocity of 3.6 m/s as it passes through its central position. When the particle is 0.2 m from the central position, it has an acceleration of 4.8 m/s^2. Calculate the amplitude and period of the oscillation.

Given: $When\ x = 0\ m, v = 3.6\ m/s$ and $when\ x = -0.2\ m, a = 4.8\ m/s^2$ (since $x = -k^2 a$)

SHM: $x = A\sin(kt + \alpha)$

Find: $|A|$ and T where $T = \frac{2\pi}{k}$

Solution:

$x = -k^2 a$ $-0.2 = -k^2(4.8)$ $k^2 = 24$ ✓ $k = 2\sqrt{6}$	$v^2 = k^2(A^2 - x^2)$ $3.6^2 = 24(A^2 - 0)$ ✓ $0.54 = A^2$ $A = \pm\frac{3\sqrt{6}}{10}$ Amplitude= $\lvert A\rvert = \frac{3\sqrt{6}}{10}$ ✓	$T = \frac{2\pi}{2\sqrt{6}}$ $T = \frac{\pi\sqrt{6}}{6} s$ ✓

Question 12 **[5 marks]**

Consider the logistic differential equation $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{200}\right)$, $P(0) = 20$.

a) Write P as a function of t.

$$\frac{dP}{dt} = gP\left(1 - \frac{P}{A}\right) \quad \Leftrightarrow \quad P = \frac{AP_0}{P_0 + (A - P_0)e^{-gt}}$$

$$P = \frac{200(20)}{20 + 180e^{-0.2t}} \quad \Rightarrow \quad P = \frac{200}{1 + 9e^{-0.2t}}$$ ✓

b) Find the value of P when $t = 10$.

$P \approx 90.2$ ✓

c) Discuss the behaviour of P as $t \to \infty$.

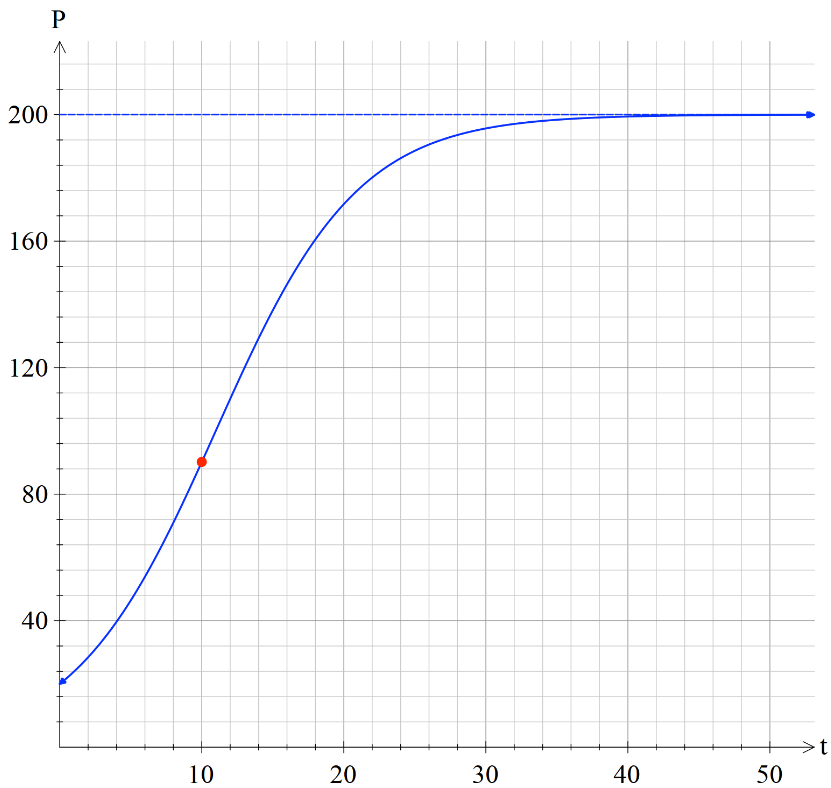
$P \to 200$ ✓ (The population approaches 200.)

No mark if the **limiting value of the logistic model** is not identified.

d) Sketch the graph of P against t.

✓ **1 mark for fairly accurate logistic curve. No other shapes accepted.**

✓ **1 mark for precise accuracy of the logistic curve with asymptote indicated.**



Question 13 **[6 marks]**

Ben has a young son who was recently ill with a fever. He noticed that after being given a dose of penicillin the child's temperature increased, peaked and then decreased. Ben approximated the child's temperature above 37° by the function $T(x) = xe^{-0.2x}$ where $x > 0$, where x refers to the time in hours after 7.00pm. Using this model find:

a) The rate of change of temperature and show that for $0 < x < 5$, the temperature was increasing.

b) The maximum temperature and the time this occurred.

a) $T'(x) = 1.e^{-0.2x} + x.(-0.2)e^{-0.2x} = e^{-0.2x}(1 - 0.2x)$

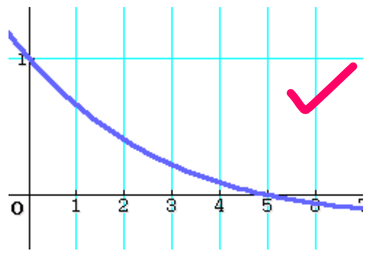
The temperature is increasing when $T'(x) > 0$

$e^{-0.2x} > 0, \forall x \epsilon \mathbb{R}$

$1 - 0.2x > 0 \;\; \Rightarrow \;\; 1 > 0.2x \;\; \Rightarrow \;\; 5 > x$

$\therefore 0 < x < 5, T'(x) > 0$

Alternately:



T'(x)>0 for 0<x<5

b) Max T when $T'(x) = 0$ *and* $T''(x) < 0$

$T'(x) = 0$ when $x = 5$ $\therefore$ *Time is* 12 *Midnight.* (12*AM*)

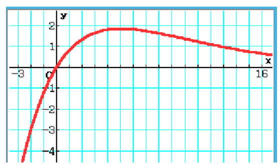
Checking max (not min)
Using derivative
Or using sketch of graph
Or referencing the increasing
Nature of temp determined in a).

$\frac{d^2}{dx^2}(x \cdot e^{-0.2 \cdot x})$

$\frac{(x-10) \cdot e^{\frac{-x}{5}}}{25}$

$\frac{(x-10) \cdot e^{\frac{-x}{5}}}{25} | x=5$

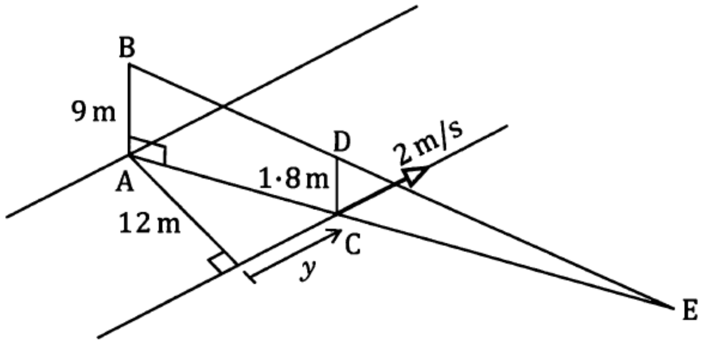
$\frac{-e^{-1}}{5}$



$T(5) = 5e^{-0.2 \times 5} = \frac{5}{e} \approx 1.84 \quad \therefore$ *Max temp is* 38.8°

Question 14 **[6 marks]**

The diagram, not shown to scale, shows a lamp post, AB, of height 9 metres, situated on one side of a straight road.



The road is of width 12 metres and line CD, see diagram, represents a person of height 1.8 metres walking at 2 metres/second along the other side of the road from the lamp. The light at B causes the person to have a shadow shown as CE. Find the rate at which the length of the shadow is changing at the instant when AC is of length 20 metres.

Let CE = x, need $\frac{dx}{dt}$, given $\frac{dy}{dt} = 2m/s$ Using similar triangles (AA): $\frac{x}{1.8} = \frac{x + AC}{9}$ ✓ $9x = 1.8x + 1.8AC$ $7.2x = 1.8AC$ $4x = AC$ ✓	$AC^2 = 144 + y^2$ $(4x)^2 = 144 + y^2$ $16x^2 = 144 + y^2$ ✓ Diff w.r.t. t: $32x\frac{dx}{dt} = 2y\frac{dy}{dt}$ ✓ $32(5)\frac{dx}{dt} = 2(16)(2)$ $\frac{dx}{dt} = 0.4\ m/s$ ✓	When AC = 20, $x = 5\ m, y = 16\ m$ ✓